

## Weak-link synchronization

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We describe a mutual synchronization mechanism observed in a model of fiber laser arrays. Though sub-optimal in terms of total coherent power, the weak-link-synchronized state is far more robust than the in-phase state of a uniformly pumped array, with respect to parameter mismatch among the individual elements. We find similar dynamical behavior in a more general system of coupled oscillators where the amplitude dynamics is crucial.

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### I. INTRODUCTION

The subject of synchronization in nonlinear oscillator arrays has received a great deal of attention [1–3]. Besides its intrinsic interest for the field of nonlinear dynamics, the goal of synchronization is important in many practical contexts. In some cases, the desired behavior can be achieved by so-called injection locking, so that (for example) a generator is entrained by a controlled external signal which is weak but very precise [1]. In contrast to this master-slave situation there is the problem of mutual synchronization among a population of nominally identical oscillators, of which the Kuramoto model is the archetype [4].

In this paper, we describe a kind of mutual synchronization which occurs in a model of high-gain and high-loss fiber lasers. Typically, the search for highly coherent output states considers schemes where the individual elements are as identical as possible and driven identically [5]. In contrast, we investigate arrays where such uniformity is intentionally avoided by driving some elements more strongly than others. We find that these inhomogeneous arrays can operate in a highly coherent way via a mechanism we call weak-link synchronization. The weak-link-synchronized states, though suboptimal compared with the uniformly pumped array, is far more robust with respect to parameter mismatch among the individual elements. The practical advantage of weak-link synchronization may therefore be especially pronounced for very large arrays.

Weak-link synchronization represents a trade-off between optimization in principle and optimization in practice. In principle, a fully symmetric coherent inphase state is ideal. In practice, it may be difficult to achieve the necessary tolerances. We find that employing a strategy of intentionally nonuniform (but patterned) driving yields an attractor with both a high degree of coherence and robustness under parameter mismatch. Our numerical simulations suggest that the scheme can be applied to very large arrays without substantial degradation.

### II. BACKGROUND

Our starting point is a model recently developed to describe coupled fiber lasers in the high-gain and high-loss limit. A detailed derivation of these may be found in Ref. [6]. For our purposes, each fiber may be considered to be a light tube with a short-gain region, with one end perfectly reflecting and the other poorly reflecting. The latter feature is responsible for describing the dynamics as an iterative map rather than a system of differential equations: the electric field may change rapidly between consecutive round-trips, so the standard slowly varying wave approximation (SVWA) is not valid [7]. The fibers interact over a short length by passing through a coupler which mixes the fields. The dynamical model describing the evolution over one round-trip of the complex electric fields  $E_n$  and the real gains  $G_n$  is [6]

$$E_n(t+T) = r \sum_{m=1}^N e^{[G_n(t)+G_m(t)]/2} S_{nm} E_m(t), \quad (1)$$

$$G_n(t+T) = G_n(t) + \epsilon [G_n^p - G_n(t) - 2(1 - e^{-G_n(t)}) |E_n(t)|^2]. \quad (2)$$

Here,  $S_{nm}$  is a coupling matrix, whose elements depend on the architecture and properties of the coupler,  $r$  is the reflection coefficient,  $G_n^p$  is the pumping parameter, and  $\epsilon$  is the ratio of the round-trip time  $T$  to the fluorescence time. In practice,  $\epsilon$  is very small (say,  $10^{-4}$ ), while  $r$  is on the order of 20%. The derivation of the dynamical model is broad enough to cover a variety of physical coupling mechanisms; in what follows, we consider coupling via classical “cross talk” between packed fibers [8].

Ultimately, one would like to identify the conditions (if any) under which very large arrays will behave in a highly coherent manner. The most common strategy is to consider a system of identical elements, identically driven and with fully symmetric coupling, and then explore the linear stability of the fully symmetric solution in which all elements behave identically [9]. We take a different approach, since

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we want to look more broadly at other types of synchronized solutions and test their sensitivity with respect to disordering effects (such as detuning).

In fact, the case of just two elements yields important insight, and our present analytic understanding of weak-link synchronization rests on a careful analysis of the  $N=2$  problem. This case yields the simplest incarnation of the weak-link idea and demonstrates the relative pros and cons of weak-link-synchronized states as compared with the in-phase states of the uniformly driven array. Later, we use this insight to design arrays with much larger  $N$  and explore these numerically.

### III. ANALYSIS FOR TWO LASERS

In this case the coupling matrix elements are [8]

$$S_{11} = S_{22}^* = \cos(p) - i \sin(q)\sin(p),$$

$$S_{12} = S_{21} = -i \cos(q)\sin(p),$$

$$q = \arctan \frac{\delta}{\kappa}, \quad p = 2z_c \sqrt{\kappa^2 + \delta^2}, \quad \delta = \frac{\beta_1 - \beta_2}{2},$$

where  $\beta_n$  is the propagation constant of the  $n$ th fiber,  $\kappa$  is the coupling constant, and  $z_c$  is the length of the coupler. The quantity  $q$  corresponds to a detuning parameter, while  $p$  is a measure of the coupling strength. In what follows, we consider the physically typical case where the latter is small.

Let  $E_n(t) = \varepsilon_n(t)e^{i\psi_n(t)}$ , where  $\varepsilon_n$  and  $\psi_n$  are real. We seek solutions for which  $\psi(t) = \psi_1(t) - \psi_2(t)$  is constant. We do not require that the lasers be pumped identically, viewing  $G_1^p$  and  $G_2^p$  as independent control parameters. Meanwhile, any intrinsic disorder in the system is due to a nonzero value of the detuning parameter  $q$ .

Consider first the limit where there is no detuning ( $q=0$ ). A straightforward (if somewhat tedious) calculation yields several constant- $\psi$  solutions. The overall situation is illustrated in Fig. 1. The first branch of solutions (see the Appendix) exists only within a narrow strip of parameter space corresponding to nearly identically pumped lasers. It is natural therefore to introduce a small parameter  $\rho$  to express these solutions: we have

$$\begin{aligned} \psi &= \arcsin[\rho/\tan p] + O(\rho^2), \\ \varepsilon_1^2 &= \varepsilon_2^2 = I + O(\rho^2), \\ G_1 &= G_0 + \rho + O(\rho^2), \\ G_2 &= G_0 - \rho + O(\rho^2), \end{aligned} \quad (3)$$

where  $\rho = (G_1^p - G_2^p)/2(1+2rI)$ ,  $I = (G^p - G_0)/2(1-r)$ ,  $G^p = (G_1^p + G_2^p)/2$ , and  $G_0 = -\ln r$ .

This is the synchronized solution corresponding to states where both lasers operate with (nearly) equal and optimal intensity and a constant phase difference. If the lasers are identically pumped, these states are the familiar in-phase and

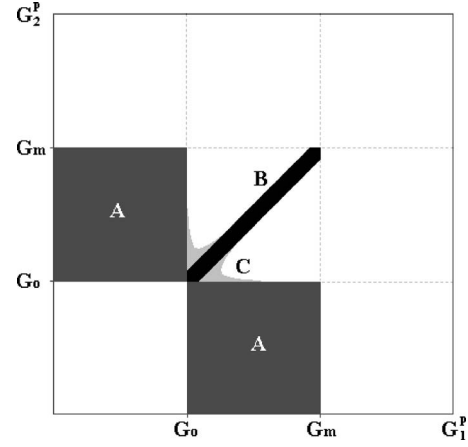


FIG. 1. Schematic summarizing the regions of stability for the various fixed point solutions: A: weak link solutions. B: well-pumped solutions, Eq. (12). C: near-threshold transitional solutions.

antiphase states  $\psi=0$  and  $\pi$ , respectively, described, e.g., for coupled semiconductor lasers in Ref. [10].

A careful analysis shows that this solution is attracting in its narrow region of existence as long as

$$0 < I < I_m \equiv 1/\sqrt{4-6r}$$

$$\left( \text{or equivalently } G_0 < G^p < G_m \equiv G_0 + \frac{1-r}{2-3r} \right)$$

and

$$p|\cos \psi| < \sqrt{\frac{\epsilon}{2} \left( 1 - \frac{I}{I_m} \right)}.$$

Now suppose we allow for a small amount of detuning ( $q \neq 0$ ) to find the corresponding corrections to this solution. Of particular importance is the correction to the relative phase, which is

$$\sin(\psi) = \frac{G_1^p - G_2^p}{2p(1+2rI)} - \frac{q(G^p - G_0)}{p(1+2rI)\cos \psi}.$$

Recall that  $p$  represents the coupling strength, which is a small quantity. It is evident that these solutions are very sensitive to parameter mismatch and might be difficult to observe in practice. All of the foregoing refers to the fixed points labeled B in Fig. 1.

Now consider another set of fixed point solutions, labeled A in Fig. 1. These are given by (in the limit of zero detuning)

$$\begin{aligned} \psi &= \pi/2, & \psi &= -\pi/2, \\ \varepsilon_1 &= \sqrt{I_1} + O(p^2), & \varepsilon_1 &= p \frac{\sqrt{rI_2 \exp G_1^p}}{1-r \exp G_1^p} + O(p^2), \\ \varepsilon_2 &= p \frac{\sqrt{rI_1 \exp G_2^p}}{1-r \exp G_2^p} + O(p^2), & \varepsilon_2 &= \sqrt{I_2} + O(p^2), \end{aligned}$$

$$G_1 = G_0 + O(p^2), \quad G_1 = G_1^p + O(p^2),$$

$$G_2 = G_2^p + O(p^2), \quad G_2 = G_0 + O(p^2).$$

The corresponding regions of stability are, respectively,

$$G_0 < G_1^p < G_m, \quad \text{and} \quad 0 < G_1^p < G_0,$$

$$0 < G_2^p < G_0, \quad G_0 < G_2^p < G_m.$$

These solutions correspond to a state where one laser is well pumped ( $G^p > G_0$ ) and operates with relatively large intensity, while the other laser is underpumped ( $G^p < G_0$ ) and has a very small amplitude (of order of  $p$ ). Even so, these are synchronized states because the two lasers maintain a constant- $\pi/2$ -phase shift. For reasons that will become clear when we turn to larger arrays, we call this the “weak-link solution.” The fact that the underpumped laser has nonzero intensity is important, because it is how phase information is transmitted across larger arrays.

Although the total intensity of the weak-link solution is less than the intensity of the “fully pumped” branch of solutions, the weak-link solution has significant benefits: (1) it has a very broad region of existence and stability, and (2) it is extraordinarily robust to parameter mismatch. Corrections to first order in  $q$  do not have any serious influence on this solution, in contrast to the fully pumped solution. In particular, the phase difference is only slightly modified:

$$\psi = \pm \left( \frac{\pi}{2} + \frac{rpq}{\exp(-G^p) - r} \right).$$

There is yet one more fixed point solution, which is labeled  $C$  in Fig. 1. However, its region of stable existence is extremely small and we do not consider it further. For completeness, we display this solution in the Appendix. It may hold some mathematical interest, in some sense providing a continuous transition between the other solution branches.

Formally, the oscillations of the underpumped laser exist only due to the interaction with another laser, but we cannot consider this system as a resonant driving of a dissipative oscillator. First of all the interaction is produced through cross talk, so that instead of direct driving, the well-pumped laser works as an energy source for small self-sustained oscillations in the underpumped laser. Second of all, the well-pumped laser is not an oscillator with a fixed frequency, but a dynamical unit which also adjusts its rhythm in response to the interaction with the weak one. This interpretation of the system behavior as the synchronization phenomenon is especially important for understanding of the dynamics of larger hybrid arrays. Indeed, if we assume that the underpumped laser is just a passive driven oscillator and does not drive back the well-pumped oscillator, then two well-pumped lasers separated by the underpumped one (i.e., indirectly coupled through the weak oscillator) would not be able to synchronize. In the next section we show that this is not true and that weak dynamical coupling (weak link) provides strong and robust synchronization.

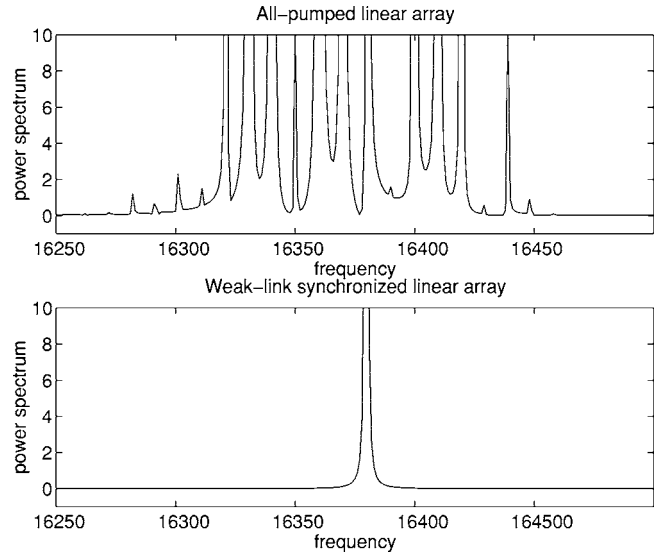


FIG. 2. Power spectrum  $P(\omega)$  for the linear array of 19 lasers with near-neighbor coupling, with  $r=0.2$ ,  $\epsilon=0.005$ ,  $\delta\sim 10^{-4}$ ,  $\kappa=0.015$ , and  $z_c=1$ . Initial conditions were chosen randomly with  $\epsilon_n \in (0, 1)$ ,  $\psi_n \in (0, 2\pi)$ , and  $G_n \in (1.6, 2.3)$ . The upper plot is for the case when all lasers are strongly pumped, with  $G^p=1.9$ . The lower plot is for the case when instead every other laser is underpumped, with  $G^p=0.4$  (weak-link synchronization).

#### IV. MANY LASERS

The situation that we have just described suggests a scheme for how one might try to synchronize a large array of lasers if the uniformly pumped configuration is too sensitive to be successfully synchronized. Suppose we have three lasers, with the outer two well pumped and the middle underpumped. If each of the well-pumped lasers easily synchronizes with the underpumped one, then they will synchronize with each other. Thus, by sacrificing the intensity of one laser, one might robustly and effectively synchronize the other two through a weak link, which would be otherwise fail.

To test this idea, we consider first a linear array of coupled fiber lasers and compare the configuration when all the lasers are well pumped against the configuration when every other laser is underpumped. For this and other large arrays to follow, we use near-neighbor coupling to generate the matrix elements  $S_{nm}$ , as described in the Appendix. As a convenient quantitative measure of the output, we use the power spectrum of the total electric field:

$$P(\omega) = \left| \int_{-\infty}^{\infty} E_{tot}(t) e^{i\omega t} dt \right|^2,$$

where

$$E_{tot} = \sum_n \epsilon_n(t) e^{i\psi_n(t)}.$$

Figure 2 shows the results for an array of  $N=19$  lasers. We have introduced a small amount of intrinsic disorder ( $\delta\sim 10^{-4}$ ). The upper panel is the result when all lasers are pumped with  $G^p=1.9$ ; the lower panel is the result when we

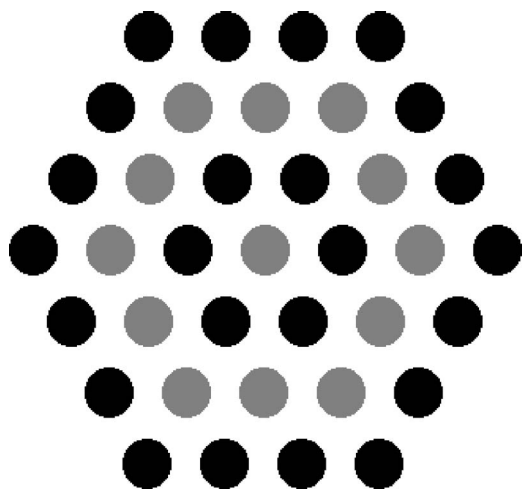


FIG. 3. Cross section of the 37-fiber system and the pumping scheme used to achieve weak-link synchronization. Black and gray circles represent the well-pumped and underpumped lasers, respectively.

reduce the pump for nine of the lasers to  $G^p=0.4$ , a value below the single-laser threshold value. The other parameters are listed in the figure caption. Repeated trials indicate that these results are independent of initial conditions. The uniformly pumped configuration does not generate a coherent output; consequently, the contrast is considerably lower than for the corresponding weak-link pumping scheme, whose power spectrum has a single sharp line.

There are other strategies one can try to get the fully pumped array to synchronize. For example, one can individually trim the pump parameters of each laser in the hope of somehow compensating for the intrinsic disorder. In fact, we were unable to get anything even approaching the clean output of the weak-link-synchronized state; there are reasons to suspect that it is impossible in practice.

Yet another strategy is to change the topology of the array—e.g., by introducing periodic boundary conditions. This strategy works: a one-dimensional ring of these lasers can robustly synchronize to a synchronized state. On the other hand, using a ring architecture begs the question of still larger arrays. After all, a large ring takes up a large cross-sectional area, most of which is empty space. From the perspective of total power, a more attractive possibility is to use a large bundle of fiber lasers. We consider, therefore, a bundle comprising four concentric rings of 1, 6, 12, and 18 lasers. The resulting arrangement of 37 lasers is illustrated in Fig. 3. In view of the robust synchronization of a single ring, we choose for our weak-link configuration one where every other ring is underpumped. We compare the output against the output of the corresponding fully pumped configuration, as shown in Fig. 4. In the well-pumped array we find no synchronization; the other array readily falls into the weak-link-synchronized attractor.

The same strategy works for even larger arrays. We have observed robust weak-link synchronization in simulations with as many as 91 elements (comprising 6 concentric rings) with no apparent degradation.

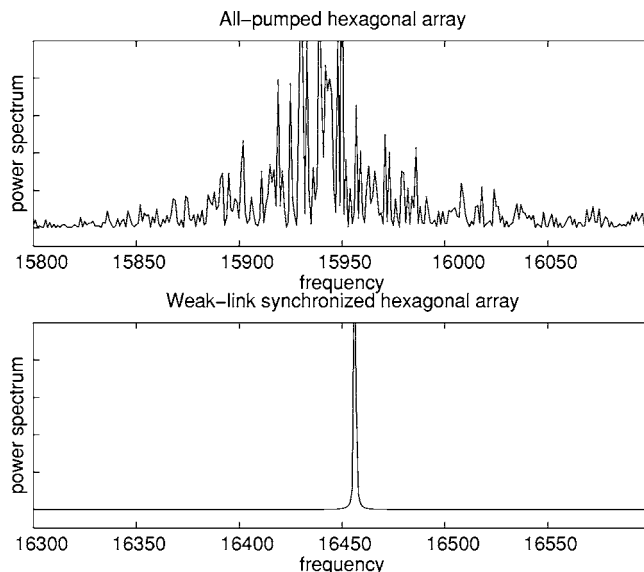


FIG. 4. Same as Fig. 2 but for the two-dimensional ( $N=37$ ) array. The upper plot is for the all-pumped configuration; the lower plot is for the weak-link arrangement.

### V. GENERAL OSCILLATORS

Is the behavior of the fiber laser array unique, or is weak-link synchronization a more general phenomenon? Most of the principal ingredients of the laser system can be observed in an archetypical model which describes a broad range of physical problems. Consider the general equations of a chain of coupled Hopf oscillators with slowly varying complex amplitudes:

$$\dot{A}_i = i\omega_i A_i + \mu_i A_i - (\gamma_i + i\alpha_i) |A_i|^2 A_i + \sum_j S_{ij} (A_j - A_i).$$

For simplicity we restrict ourselves to nearest-neighbor coupling  $S_{ij} = i\kappa(\delta_{j,i+1} + \delta_{j,i-1})$ , with the same nonlinear frequency shift  $\alpha_i/\gamma_i \equiv \alpha$  for all oscillators. Introducing real amplitudes and phases according to  $A_i = (r_i/\sqrt{\gamma_i})e^{i\theta_i}$ , we get

$$\dot{r}_i = \mu_i r_i - r_i^3 + \kappa r_{i-1} \sin(\theta_i - \theta_{i-1}) - \kappa r_{i+1} \sin(\theta_{i+1} - \theta_i),$$

$$\begin{aligned} \dot{\theta}_i = & \omega_i - \alpha r_i^2 - 2\kappa + \kappa \frac{r_{i-1}}{r_i} \cos(\theta_i - \theta_{i-1}) + \kappa \frac{r_{i+1}}{r_i} \\ & \times \cos(\theta_{i+1} - \theta_i). \end{aligned}$$

Note that without coupling ( $\kappa=0$ ), each oscillator obeys the normal form equation for a supercritical Hopf bifurcation: when  $\mu < 0$  the origin  $r=0$  is a stable spiral; when  $\mu > 0$ , there is an unstable spiral at the origin and a stable circular limit cycle at  $r = \sqrt{\mu}$ .

For  $N=2$  these equations show similar behavior to two fiber lasers: if both oscillators are “turned on” (i.e.,  $\mu_{1,2} > 0$ ), then they synchronize to each other in terms of frequency entraining and constant phase difference only if the detuning  $\Delta\omega = |\omega_1 - \omega_2|$  and mismatch  $\Delta\mu = |\mu_1 - \mu_2|$  are small enough, but if one of them is turned off (so  $\mu_1 < 0$  and  $\mu_2 > 0$ ), then they synchronize regardless of  $\Delta\omega$  and  $\Delta\mu$ , with the first

oscillator having very small (but nonzero) amplitude.

What happens to a linear array of these oscillators with every other one underpumped—i.e., taking  $\mu_{2i} < 0$  and  $\mu_{2i+1} > 0$  and some distribution of  $\omega_i$ ? Such a system indeed synchronizes, though there are significant differences with the laser system. First of all, even in the case of the weak-link configuration the Hopf array is still sensitive to the width of the natural frequency distribution,  $\Delta\omega$ . Moreover, for some small  $\Delta\omega$  the configuration of all-turned-on oscillators with close positive  $\mu$ 's also readily synchronizes, which was not the case for the fiber lasers. This means that although weak-link synchronization occurs in the Hopf array, it does not provide the obvious benefit of superior robustness when compared against the conventional strategy of using uniformly pumped arrays.

That said, we see that weak-link synchronization is not a peculiar property of the fiber-laser system and may provide a useful alternative scheme for synchronizing other nonlinear oscillator arrays where conventional synchronization is troublesome or elusive.

#### ACKNOWLEDGMENTS

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#### APPENDIX

##### Fixed-point solutions

We provide details leading to the fixed-point solutions quoted in the main text. For  $N=2$ , with  $E_n = \epsilon_n e^{i\psi_n}$ , we seek solutions of Eqs. (1) and (2) with  $\epsilon_n(t+T) = \epsilon(t)$ ,  $\psi_1(t+T) - \psi_2(t+T) = \psi_1(t) - \psi_2(t) = \psi$ , and  $G_n(t+T) = G_n(t)$ . It follows that

$$\epsilon_1 \epsilon_2 \cos \psi = r^2 e^{G_1+G_2} \epsilon_1 \epsilon_2 \cos \psi, \quad (\text{A1})$$

$$\begin{aligned} \epsilon_1 \epsilon_2 \sin \psi &= r^2 e^{G_1+G_2} \epsilon_1 \epsilon_2 (\cos^2 p - \sin^2 p) \sin \psi \\ &\quad + r e^{(G_1+G_2)/2} (r e^{G_1} \epsilon_1^2 - r e^{G_2} \epsilon_2^2) \sin p \cos p, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \epsilon_1^2 &= r^2 e^{2G_1} \epsilon_1^2 \cos^2 p + r^2 e^{G_1+G_2} \epsilon_2^2 \sin^2 p \\ &\quad - 2r^2 e^{(3G_1+G_2)/2} \epsilon_1 \epsilon_2 \sin p \cos p \sin \psi, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \epsilon_2^2 &= r^2 e^{2G_2} \epsilon_2^2 \cos^2 p + r^2 e^{G_1+G_2} \epsilon_1^2 \sin^2 p \\ &\quad + 2r^2 e^{(G_1+3G_2)/2} \epsilon_1 \epsilon_2 \sin p \cos p \sin \psi, \end{aligned} \quad (\text{A4})$$

$$G_1 = G_1 + \epsilon [G_1^p - G_1 - 2(1 - e^{-G_1}) \epsilon_1^2], \quad (\text{A5})$$

$$G_2 = G_2 + \epsilon [G_2^p - G_2 - 2(1 - e^{-G_2}) \epsilon_2^2]. \quad (\text{A6})$$

We demand that  $\epsilon_1$ ,  $\epsilon_2$ ,  $G_1$ , and  $G_2$  be strictly positive; then from Eq. (A1),

$$\cos(\psi) [1 - r^2 e^{G_1+G_2}] = 0,$$

which gives two branches of solutions: (1)  $r^2 e^{G_1+G_2} = 1$  and (2)  $\cos(\psi) = 0$ . Substituting the first expression into Eqs. (A2)–(A4) yields

$$\epsilon_1 = \epsilon_2, \quad (\text{A7})$$

$$\sin(\psi) = \frac{r(e^{G_1} - e^{G_2})}{2 \tan p}. \quad (\text{A8})$$

But since  $|\sin(\psi)| \leq 1$  and  $p \ll 1$ , then  $|e^{G_1} - e^{G_2}| \leq 2 \tan p / r \ll 1$ . On the other hand, from Eqs. (A5)–(A7) one finds that  $G_1 = G_2$  if and only if  $G_1^p = G_2^p$ , whence the solution is

$$\psi = 0, \pi, \quad \epsilon_1 = \epsilon_2 = \sqrt{\frac{G^p + \ln r}{2(1-r)}}, \quad G_1 = G_2 = -\ln r.$$

This solution exists only for small  $|G_1^p - G_2^p|$ , and we can introduce a small parameter  $\rho = (G_1^p - G_2^p)/2(1+2rI)$ , so that

$$\psi = \arcsin[\rho/\tan p] + O(\rho^2),$$

$$\epsilon_1^2 = \epsilon_2^2 = I + O(\rho^2),$$

$$G_1 = G_0 + \rho + O(\rho^2),$$

$$G_2 = G_0 - \rho + O(\rho^2), \quad (\text{A9})$$

where  $I = (G^p - G_0)/2(1-r)$ ,  $G^p = (G_1^p + G_2^p)/2$ , and  $G_0 = -\ln r$ .

Meanwhile, the branch of solutions in region C of Fig. 1 is given by (for zero detuning)

$$\psi = \pm \frac{\pi}{2},$$

$$\begin{aligned} \epsilon_1 &= \sqrt{\frac{G_1^p - G_0}{2(1-r)}} \left[ 1 \mp \frac{p}{2} \sqrt{\frac{G_2^p - G_0}{G_1^p - G_0}} \left( \frac{1}{G_1^p - G_0} + \frac{r}{1-r} \right) \right] \\ &\quad + O(p^2), \end{aligned}$$

$$\begin{aligned} \epsilon_2 &= \sqrt{\frac{G_2^p - G_0}{2(1-r)}} \left[ 1 \pm \frac{p}{2} \sqrt{\frac{G_1^p - G_0}{G_2^p - G_0}} \left( \frac{1}{G_2^p - G_0} + \frac{r}{1-r} \right) \right] \\ &\quad + O(p^2), \end{aligned}$$

$$G_1 = G_0 \pm p \sqrt{\frac{G_2^p - G_0}{G_1^p - G_0}} + O(p^2),$$

$$G_2 = G_0 \mp p \sqrt{\frac{G_1^p - G_0}{G_2^p - G_0}} + O(p^2),$$

with the first-order correction to the phase difference given by

$$\psi = \pm \frac{\pi}{2} \pm 2q \frac{\sqrt{(G_1^p - G_0)(G_2^p - G_0)}}{G_2^p - G_1^p}.$$

#### 2. Coupling matrix

To determine the coupling matrix  $S_{nm}$  for arrays (i.e.,  $N > 2$ ), we integrated the equation

$$\frac{dE_n}{dz} = iA_{nm}E_m,$$

where  $z$  is the spatial coordinate along the fiber,  $A_{nm} = \beta_n \delta_{nm} + C(d_{nm})$ , and  $d_{nm}$  is the distance between the centers of the  $n$ th and  $m$ th fibers. Integrating over the total coupling length  $z_c$  yields  $E_n(z=z_c) = S_{nm}E_m(z=0)$ . Typically, we take the nearest-neighbor coupling

$$C(d_{nm}) = \begin{cases} 0 & \text{if } d_{nm} > 2R, \\ \kappa & \text{if } d_{nm} = 2R, \end{cases}$$

where  $R$  is the radius of each fiber. Our results are essentially unchanged for other functions  $C(d_{nm})$  provided they decay fast enough.

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